

SINGULAR INTEGRALS AND MAXIMAL FUNCTIONS ASSOCIATED WITH HIGHLY MONOTONE CURVES

BY

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ABSTRACT. Let $\gamma: [-1, 1] \rightarrow \mathbf{R}^n$ be an odd curve. Set

$$H_\gamma f(x) = \text{PV} \int f(x - \gamma(t)) \frac{dt}{t}$$

and

$$M_\gamma f(x) = \sup h^{-1} \int_0^h |f(x - \gamma(t))| dt.$$

We introduce a class of highly monotone curves in $\mathbf{R}^n$, $n > 2$, for which we prove that H_γ and M_γ are bounded operators on $L^2(\mathbf{R}^n)$. These results are known if γ has nonzero curvature at the origin, but there are highly monotone curves which have no curvature at the origin.

Related to this problem, we prove a generalization of van der Corput's estimate of trigonometric integrals.

Introduction. Let $\gamma: [-1, 1] \rightarrow \mathbf{R}^n$ be an odd continuous curve. For a test function f on $\mathbf{R}^n$ we define the "Hilbert transform along γ " of f by

$$(H_\gamma f)(x) = \text{PV} \int_{-1}^1 f(x - \gamma(t)) \frac{dt}{t}$$

and the "maximal function along γ " of f by

$$(M_\gamma f)(x) = \sup_{0 < \varepsilon < 1} \frac{1}{\varepsilon} \int_0^\varepsilon |f(x - \gamma(t))| dt.$$

We seek conditions on γ which guarantee either of the estimates

(1) for each $f \in L^p(\mathbf{R}^n)$, $\|H_\gamma f\|_p \leq C_\gamma \|f\|_p$, or

(2) for each $f \in L^p(\mathbf{R}^n)$, $\|M_\gamma f\|_p \leq C_\gamma \|f\|_p$,

for some p , $1 < p < \infty$.

The operators H_γ arise when one applies the method of rotations to nonisotropic Calderón-Zygmund operators. Thus estimates of H_γ lead to estimates of a broad class of singular integrals. See Nagel, Rivière and Wainger [1], for example.

M_γ is the maximal operator naturally related to H_γ . The estimate (2) implies the following theorem on differentiation of integrals.

(3) If f is locally in $L^p(\mathbf{R}^n)$, then

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^\varepsilon f(x - \gamma(t)) dt = f(x) \quad \text{a.e.}$$

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(1), (2) and (3) are false for arbitrary C^∞ curves. See Nagel and Wainger [4, Theorem 4] and Stein and Wainger [8] for the counterexamples.

We call a C^∞ curve $\gamma: [-1, 1] \rightarrow \mathbf{R}^n$, $\gamma(0) = 0$, *well-curved* if for some $\varepsilon > 0$, $\gamma([- \varepsilon, \varepsilon])$ lies in the linear span of $\{\gamma^{(k)}(0): k = 1, 2, 3, \dots\}$. Stein and Wainger [8], [9] have shown that (1) and (2) hold if γ is well-curved and $1 < p < \infty$.

Let $\gamma(t) = (t, \phi(t))$ be a plane curve which satisfies

(4) ϕ is odd, $\phi \in C^2([0, 1])$, $\phi(0) = \phi'(0) = 0$ and $\phi'' \geq 0$ and ϕ'' is increasing on $[0, 1]$.

Note that $\phi(t) = \operatorname{sgn}(t)\exp(-|t|^{-1})$ satisfies (4), but that γ is not well-curved. Nagel and Wainger [4] have shown that (1) holds for $\frac{2}{3} < p < \frac{5}{2}$ and that (1) may fail if $p = 2$ and ϕ'' is not monotone. Stein and Wainger [9, p. 1292] have shown that (2) holds for $2 \leq p \leq \infty$.

In this paper we introduce a class of "highly monotone" curves in $\mathbf{R}^n$ which reduces to the curves in (4) when $n = 2$. The precise definition of this class can be found in §2. In §§3 and 4 we prove the following theorems under the hypothesis that $\gamma: [-1, 1] \rightarrow \mathbf{R}^n$ is odd, $\gamma_1(t) = t$ and γ is highly monotone on $[0, 1]$.

THEOREM 1. $\|H_\gamma f\|_2 \leq C_n \|f\|_2$ ($f \in L^2(\mathbf{R}^n)$).

THEOREM 2. $\|M_\gamma f\|_p \leq C_{n,p} \|f\|_p$ ($f \in L^p(\mathbf{R}^n)$, $2 < p < \infty$).

THEOREM 3. If f is locally in $L^p(\mathbf{R}^n)$, $2 < p < \infty$, then

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^\varepsilon f(x - \gamma(t)) dt = f(x) \quad a.e.$$

At this point we wish to say a few words about the proofs of Theorems 1 and 2.

Theorem 1 is equivalent to the boundedness of the Fourier multiplier

$$m_\gamma(\xi) = \operatorname{PV} \int_{-1}^1 e^{-2\pi i \xi \cdot \gamma(t)} \frac{dt}{t}.$$

In [7] Stein and Wainger introduced the following estimate of trigonometric integrals into the study of H_γ .

LEMMA (VAN DER CORPUT). If f is a real-valued function on $[a, b]$, if $|f^{(n)}(t)| \geq \lambda > 0$ for $a \leq t \leq b$ and if f' is monotone when $n = 1$, then

$$\left| \int_a^b e^{if(t)} dt \right| \leq C_n \lambda^{-1/n}.$$

This lemma has been basic in the study of H_γ and M_γ since then.

In §1 we prove a generalization of van der Corput's lemma which is particularly suited to highly monotone curves. The generalization comes in replacing $f^{(n)}$ by $D^n f$ when $n > 1$, where D^n is a suitable differential operator of order n . In §3 we use this estimate to prove Theorem 1 for highly monotone curves.

To prove Theorem 2 we use the method of "g-functions" introduced by E. M. Stein in [5] and [6]. This technique has been used to prove many maximal theorems; for examples see Stein and Wainger [8], Nagel, Stein and Wainger [3] and Wainger [10].

The use of g -functions allows one to use the Fourier transform and reduce the maximal theorem to estimating trigonometric integrals. In §4 we use a variant of the g -function in Stein and Wainger [9, p. 1292] and our generalization of van der Corput's lemma to prove Theorem 2 for highly monotone curves.

It should be noted that the first application of the Fourier transform to the study of M_γ was made by Nagel, Rivière and Wainger [2] in the special case $\gamma(t) = (t, t^2)$.

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1. An estimate for trigonometric integrals. In this section, we prove a lemma which will be of use in §§3 and 4. Before stating the lemma we must introduce some notation.

Given a smooth function $\alpha: [a, b] \rightarrow (0, \infty)$ we define a differential operator D_α by

$$D_\alpha f(t) = (f/\alpha)'(t).$$

If $\alpha_1, \dots, \alpha_n$ are n such functions, then we inductively define operators $D^1, \dots, D^n$,

$$D^1 = D_{\alpha_1}; \quad D^{k+1} = D_{\alpha_{k+1}} D^k \quad \text{for } 1 \leq k < n.$$

LEMMA 1. *Let $\alpha_1, \dots, \alpha_n$ be positive nondecreasing functions defined on $[a, b]$ and let $\alpha_1 \equiv 1$. Let f be a real-valued function of class C^n on $[a, b]$. If $D^1 f = f'$ is monotone and if $|D^n f(t)| \geq \lambda > 0$ for each t in $[a, b]$, then*

$$\left| \int_a^b \exp[if(t)] dt \right| \leq C_n (\lambda \alpha_1(a) \cdots \alpha_n(a))^{-1/n}.$$

We note that if each $\alpha_k \equiv 1$, then Lemma 1 reduces to van der Corput's lemma. (See [11, Volume I, p. 197] or [9, p. 1258].)

PROOF. Suppose that $n = 1$. To be specific, assume that f' is nonincreasing and that $f'(t) \geq \lambda$. Now

$$\int_a^b e^{if(t)} dt = \int_a^b \frac{d}{dt} \{ e^{if(t)} \} \frac{dt}{if'(t)}$$

so integration by parts yields the estimate

$$\left| \int_a^b e^{if(t)} dt \right| \leq \frac{1}{f'(b)} + \frac{1}{f'(a)} + \int_a^b d\left(\frac{1}{f'}\right) = \frac{2}{f'(b)} < \frac{2}{\lambda}.$$

Now assume that the lemma is true for a given $n \geq 1$. Assume that f' is monotone and that $D^{n+1} f(t) \geq \lambda$ for $a \leq t \leq b$.

Set $h = D^n f$. Then $D^{n+1} f = (h/\alpha_{n+1})'$. Choose c in $[a, b]$ so that h/α_{n+1} is positive on (c, b) and h/α_{n+1} is negative on (a, c) . Such a value of c exists, and is unique, since h/α_{n+1} is increasing.

Write $\int_a^b e^{if(t)} dt = \int_a^c + \int_c^b = P + Q$ and estimate P and Q separately. To estimate P suppose that $a < u < c$. Then

$$|P| \leq \left| \int_a^u e^{if(t)} dt \right| + c - u.$$

If $a \leq t \leq u$, then

$$\frac{h}{\alpha_{n+1}}(t) \leq \frac{h}{\alpha_{n+1}}(u) = \frac{h}{\alpha_{n+1}}(c) - \int_u^c \left(\frac{h}{\alpha_{n+1}} \right)'(s) ds \leq -\lambda(c - u).$$

Hence we have

$$D^n f(t) = h(t) \leq -\lambda \alpha_{n+1}(t)(c - u) \leq -\lambda \alpha_{n+1}(a)(c - u)$$

for $a \leq t \leq u$. By the induction hypothesis

$$|P| \leq C_n (\lambda \alpha_1(a) \cdots \alpha_n(a) \alpha_{n+1}(a)(c - u))^{-1/n} + c - u$$

for $a < u < c$. This estimate actually holds for each $u < c$ since $|P| \leq c - a$.

Set $c - u = (\lambda \alpha_1(a) \cdots \alpha_{n+1}(a))^{-1/(n+1)}$. Then we get

$$|P| \leq (C_n + 1)(\lambda \alpha_1(a) \cdots \alpha_{n+1}(a))^{-1/(n+1)}.$$

The estimate of Q is made in a similar manner. Q.E.D.

2. Highly monotone curves. Let $\gamma: [0, N] \rightarrow \mathbb{R}^n$ be a curve of class C^n with $\gamma(0) = 0$. We inductively define functions $\alpha_1, \dots, \alpha_n$ as follows.

$$\alpha_1 \equiv 1; \quad \alpha_{k+1} = D^k \gamma_k \quad \text{for } 1 \leq k < n.$$

Here $D^1, \dots, D^n$ are the differential operators associated with $\alpha_1, \dots, \alpha_n$ as in §1. At each stage of this definition we must assume that α_k is positive on $(0, N)$ so that the operator D^k is well defined.

We now consider the matrix $W_\gamma = [D^k \gamma_j]_{1 \leq k, j \leq n}$. It is easy to see that W_γ is upper triangular:

$$W_\gamma = \begin{bmatrix} D^1 \gamma_1 & D^1 \gamma_2 & \cdots & D^1 \gamma_n \\ & D^2 \gamma_2 & \cdots & D^2 \gamma_n \\ & & \ddots & \vdots \\ & & & D^n \gamma_n \end{bmatrix}.$$

This follows since row $k+1$ of W_γ is obtained by dividing row k by $D^k \gamma_k$ and differentiating.

We say that γ is *highly monotone* if it has the following two properties.

- (1) If $1 \leq k \leq j \leq n$, then $D^k \gamma_j$ is positive and nondecreasing on $(0, N)$.
- (2) If $1 \leq k < j \leq n$, then $D^k \gamma_j(t) = o(D^k \gamma_k(t))$ as $t \rightarrow 0+$.

LEMMA 2. Let $\gamma: [0, N] \rightarrow \mathbb{R}^n$ be a highly monotone curve. If $1 \leq k \leq j \leq n$ and $0 < t < N$, then

$$D^k \gamma_j(t) \geq \gamma_j(t) / t^k \alpha_1(t) \cdots \alpha_k(t). \quad (i)$$

PROOF. We note that if $\varphi \in C^1([0, N])$, $\varphi(0) = 0$ and φ' is nondecreasing, then we have

$$\varphi(t) \leq t\varphi'(t). \quad (\text{ii})$$

Fix j . We prove (i) by induction on k .

$$D^1\gamma_j(t) = \gamma_j'(t) \geq \gamma_j(t)/t = \gamma_j(t)/t\alpha_1(t)$$

follows from (ii) with $\varphi = \gamma_j$ and the fact that $\alpha_1 \equiv 1$.

Assume (i) for some k , $1 \leq k < j$. Then we have

$$D^{k+1}\gamma_j(t) = \left(\frac{D^k\gamma_j}{D^k\gamma_k} \right)'(t) \geq \frac{D^k\gamma_j(t)}{t\alpha_{k+1}(t)} \geq \frac{\gamma_j(t)}{t^{k+1}\alpha_1(t) \cdots \alpha_k(t)\alpha_{k+1}(t)}.$$

The first inequality follows from (ii) with $\varphi = D^k\gamma_j/D^k\gamma_k$ and from the fact that $\alpha_{k+1} = D^k\gamma_k$. The second inequality follows from the induction hypothesis. Q.E.D.

We conclude this section with some examples of highly monotone curves.

EXAMPLE 1. Let $\gamma(t) = (t^{a_1}, \dots, t^{a_n})$ for $t \geq 0$. Suppose that $a_1 > 1$ and $a_{j+1} > a_j + 1$. Then γ is highly monotone and

$$W_\gamma = \begin{bmatrix} a_1 t^{a_1-1} & a_2 t^{a_2-1} & \cdots & a_n t^{a_n-1} \\ & c t^{a_2-a_1-1} & \cdots & c t^{a_n-a_2-1} \\ & & \ddots & \vdots \\ & & & c t^{a_n-a_{n-1}-1} \end{bmatrix}.$$

The various constants c are positive and depend only on $a_1, \dots, a_n$.

EXAMPLE 2. Let $n = 2$ and let $\gamma(t) = (t, \varphi(t))$. Then γ is highly monotone if, and only if, $\varphi(0) = 0$, $\varphi'(0) = 0$ and φ'' is positive and nondecreasing on $(0, N)$.

$$W_\gamma = \begin{bmatrix} 1 & \varphi' \\ 0 & \varphi'' \end{bmatrix}.$$

EXAMPLE 3. Let $\gamma(t) = (t, t^{a_2}e^{-t^{-\beta_2}}, \dots, t^{a_n}e^{-t^{-\beta_n}})$ for $t \geq 0$. We assume that $a_{j+1} > a_j + 1$ and $\beta_n \geq \beta_{n-1} \geq \dots \geq \beta_2 > 0$. For N sufficiently small γ is highly monotone on $[0, N]$.

3. The Hilbert transform. Let $\gamma: [-N, N] \rightarrow \mathbf{R}^n$ be a continuous curve with $\gamma(0) = 0$. For $0 < \varepsilon < N$ we define the truncated Hilbert transform by

$$H_{\varepsilon, N}f(x) = \int_{\varepsilon < |t| < N} f(x - \gamma(t)) \frac{dt}{t}.$$

In [9, p. 1284] it is shown that $H_{\varepsilon, N}f$ is a well-defined measurable function if f is locally integrable on $\mathbf{R}^n$.

THEOREM 1. Let $\gamma: [0, N] \rightarrow \mathbf{R}^n$ be a highly monotone curve with $\gamma_1(t) = t$. For $0 < t \leq N$ define $\gamma(-t) = -\gamma(t)$. There exists a constant C_n , which depends only on n , so that if $0 < \varepsilon < N$ and $f \in L^2(\mathbf{R}^n)$, then

$$\|H_{\varepsilon, N}f\|_2 \leq C_n \|f\|_2. \quad (\text{i})$$

Furthermore, for each $f \in L^2(\mathbf{R}^n)$, $H_N f = \lim_{\epsilon \rightarrow 0+} H_{\epsilon, N} f$ exists in the L^2 norm and

$$\|H_N f\|_2 \leq C_n \|f\|_2. \quad (\text{ii})$$

The case $n = 2$ of Theorem 1 is due to Nagel and Wainger [4].

Let γ be the curve in Example 3 of §2. Then γ has "no curvature" at the origin, i.e. $\gamma^{(k)}(0) = 0$ for $k = 2, 3, \dots$, $\gamma([0, N])$ lies in no proper subspace of $\mathbf{R}^n$, but, by Theorem 1, the Hilbert transform associated with γ is bounded on $L^2(\mathbf{R}^n)$. This is in contrast to the work of Stein and Wainger [9, p. 1261, Theorem 3(B)] and the counterexample of Nagel and Wainger [4, Theorem 4.1].

PROOF OF THEOREM 1. An easy computation shows that $(H_{\epsilon, N} f)^\wedge = m_{\epsilon, N} \hat{f}$ where $\hat{\cdot}$ denotes the Fourier transform and

$$\begin{aligned} m_{\epsilon, N}(\xi) &= \int_{\epsilon < |t| \leq N} \exp[-2\pi i \xi \circ \gamma(t)] \frac{dt}{t} \\ &= -2i \int_{\epsilon}^N \sin(2\pi \xi \circ \gamma(t)) \frac{dt}{t}. \end{aligned}$$

Conclusion (i) is equivalent to the estimate

$$|m_{\epsilon, N}(\xi)| \leq C_n \quad \text{for } \xi \in \mathbf{R}^n \text{ and } 0 < \epsilon < N. \quad (\text{iii})$$

We prove (iii) by induction on n . The case $n = 1$ is well known.

Let $n > 1$ and assume that (iii) holds for $n - 1$. Let $\gamma: [-N, N] \rightarrow \mathbf{R}^n$ satisfy the hypotheses of Theorem 1. Set $\bar{\gamma} = (\gamma_1, \dots, \gamma_{n-1})$ so that $\gamma = (\bar{\gamma}, \gamma_n)$. Take $\xi \in \mathbf{R}^n$, $\xi = (\bar{\xi}, \xi_n)$. If $\xi_n = 0$, then $|m_{\epsilon, N}(\xi)| \leq C_n$ follows from the induction hypothesis applied to $\bar{\gamma}$. So assume that $\xi_n \neq 0$. Define a to be the solution of $|\xi_n| \gamma_n(a) = 1$, $0 < a < N$, if it exists. Otherwise set $a = N$. (Recall $\gamma_n(0) = 0$ and γ_n is increasing. Thus $a = N$ iff $|\xi_n| \gamma_n(N) \leq 1$.)

$$\begin{aligned} \frac{-1}{2i} m_{\epsilon, N}(\xi) &= \int_{\epsilon}^a \{ \sin(2\pi \xi \circ \gamma(t)) - \sin(2\pi \bar{\xi} \circ \bar{\gamma}(t)) \} \frac{dt}{t} \\ &\quad + \int_{\epsilon}^a \sin(2\pi \bar{\xi} \circ \bar{\gamma}(t)) \frac{dt}{t} + \int_a^N \sin(2\pi \xi \circ \gamma(t)) \frac{dt}{t} \\ &= P + Q + R. \end{aligned}$$

$|Q| \leq C_n$ follows from the induction hypothesis.

$$\begin{aligned} |P| &\leq 2\pi |\xi_n| \int_0^a \gamma_n(t) \frac{dt}{t} \leq 2\pi |\xi_n| \int_0^a \gamma'_n(t) dt \\ &= 2\pi |\xi_n| \gamma_n(a) \leq 2\pi. \end{aligned}$$

If $a = N$, then $R = 0$. So assume that $0 < a < N$. By the Second Mean Value Theorem,

$$R = \frac{1}{a} \int_a^b \sin(2\pi \xi \circ \gamma(t)) dt \quad \text{for some } b, a < b < N.$$

Set $f(t) = \xi \circ \gamma(t)$. Then $|D^n f(t)| = |\xi_n| (D^n \gamma_n)(t) > |\xi_n| (D^n \gamma_n)(a)$ for $a < t < N$. So Lemma 1 will imply

$$|R| \leq C_n a^{-1} (|\xi_n| (D^n \gamma_n)(a) \alpha_1(a) \cdots \alpha_n(a))^{-1/n}$$

if we can divide $[a, N]$ into a bounded (in a and ξ) number of subintervals on each of which f' is monotone. Applying Lemma 2 we see that

$$|\xi_n| a^n (D^n \gamma_n)(a) \alpha_1(a) \cdots \alpha_n(a) > |\xi_n| \gamma_n(a) = 1$$

so that $|R| < C_n$. Thus it suffices to prove (iv) below.

(iv) $\forall \xi \in \mathbf{R}^n, f''(t) = \xi \circ \gamma''(t)$ has at most $n - 2$ zeros in $(0, N)$.

Note that $f(t) = \xi_1 t + \sum_{j=2}^n \xi_j \gamma_j(t)$ so that

$$f''(t) = \sum_{j=2}^n \xi_j \gamma_j''(t) \quad \text{and} \quad D^k f(t) = \sum_{j=k}^n \xi_j D^k \gamma_j(t)$$

for $2 \leq k \leq n$. $D^n f(t) = \xi_n D^n \gamma_n(t)$ has no zero in $(0, N)$. Since $D^n f = (D^{n-1} f / \alpha_n)'$, Rolle's theorem implies that $D^{n-1} f / \alpha_n$ has at most one zero in $(0, N)$. But $\alpha_n > 0$. Thus $D^{n-1} f$ has at most one zero in $(0, N)$. Repeating this argument $n - 2$ times shows that $f'' = D^2 f$ has at most $n - 2$ zeros in $(0, N)$. This proves (iv) and with it (i).

(ii) follows from (i) and the fact that $H_N f = \lim_{\epsilon \rightarrow 0+} H_{\epsilon, N} f$ exists in L^2 if $f \in C_c^1(\mathbf{R}^n)$. Q.E.D.

Let $\gamma(t) = \text{sgn } t(|t|^{a_1}, \dots, |t|^{a_n})$ where $a_1 \geq 1$ and $a_{j+1} > a_j + 1$ for $1 \leq j < n$. The proof of Theorem 1(i) goes through for this γ with a few minor changes. This is essentially the proof given by Stein and Wainger in [7].

4. The maximal function. Let $\gamma: [0, N] \rightarrow \mathbf{R}^n$ be a continuous curve with $\gamma(0) = 0$. We define the maximal function by

$$Mf(x) = \sup_{0 < \epsilon < N} \frac{1}{\epsilon} \int_0^\epsilon |f(x - \gamma(t))| dt.$$

In [9, p. 1284] it is shown that Mf is a well-defined measurable function if f is locally integrable on $\mathbf{R}^n$.

THEOREM 2. Let $\gamma: [0, N] \rightarrow \mathbf{R}^n$ be a highly monotone curve with $\gamma_1(t) = t$.

(i) There is a constant C_n , which depends only on n , such that for each $f \in L^2(\mathbf{R}^n)$, $\|Mf\|_2 \leq C_n \|f\|_2$.

(ii) If $2 \leq p \leq \infty$, there is a constant $C_{n,p}$ such that for each $f \in L^p(\mathbf{R}^n)$, $\|Mf\|_p \leq C_{n,p} \|f\|_p$.

THEOREM 3. Let $\gamma: [0, N] \rightarrow \mathbf{R}^n$ be a highly monotone curve with $\gamma_1(t) = t$. If f is locally in $L^p(\mathbf{R}^n)$, $2 \leq p \leq \infty$, then

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_0^\epsilon f(x - \gamma(t)) dt = f(x) \quad \text{a.e.}$$

The case $n = 2$ of Theorem 2 is due to Stein and Wainger [9, p. 1292].

The curve in Example 3 of §2 has "no curvature" at the origin, lies in no proper subspace of $\mathbf{R}^n$, but the L^2 differentiation theorem (Theorem 3) is true for this curve. This is in contrast to the work of Stein and Wainger [8].

PROOF OF THEOREM 2. We prove (i) by induction on n .

If $n = 1$, then Mf is the Hardy-Littlewood maximal function and (i) is well known in this case.

Let $n > 1$ be given and assume that (i) holds for any highly monotone curve $\bar{\gamma}$ in $\mathbf{R}^{n-1}$. Let γ be a highly monotone curve in $\mathbf{R}^n$. Set $\bar{\gamma} = (\gamma_1, \dots, \gamma_{n-1})$. Then $\bar{\gamma}$ is highly monotone and $\gamma = (\bar{\gamma}, \gamma_n)$.

We set

$$M_h f(x) = \frac{1}{h} \int_h^{2h} f(x - \gamma(t)) dt$$

and

$$N_h f(x) = \frac{1}{h} \int_h^{2h} \frac{1}{\gamma_n(h)} \int_0^{\gamma_n(h)} f(\bar{x} - \bar{\gamma}(t), x_n - s) ds dt.$$

We define

$$g(f)(x) = \left\{ \int_0^{N/2} |M_h f(x) - N_h f(x)|^2 \frac{dh}{h} \right\}^{1/2} \quad \text{for } f \in L^2.$$

The argument in Stein and Wainger [9, p. 1265] shows that if $f > 0$, then

$$Mf(x) \leq C \left(g(f)(x) + \sup_{0 < 2h < N} N_h f(x) \right).$$

So it suffices to prove the following two estimates.

$$\forall f \in L^2(\mathbf{R}^n), \quad \left\| \sup_{h>0} |N_h f| \right\|_2 \leq C_n \|f\|_2 \quad (\text{ii})$$

$$\forall f \in L^2(\mathbf{R}^n), \quad \|g(f)\|_2 \leq C_n \|f\|_2. \quad (\text{iii})$$

$$\begin{aligned} \int_{\mathbf{R}^n} \sup_h |N_h f(x)|^2 dx &= \int_{\mathbf{R}^1} \int_{\mathbf{R}^{n-1}} \sup_h |N_h f(x)|^2 d\bar{x} dx_n \\ &\leq \int_{\mathbf{R}^1} C_n \int_{\mathbf{R}^{n-1}} \left(\sup_{\varepsilon>0} \frac{1}{\varepsilon} \int_0^\varepsilon |f(\bar{x}, x_n - s)| ds \right)^2 d\bar{x} dx_n \\ &= C_n \int_{\mathbf{R}^{n-1}} \int_{\mathbf{R}^1} \left(\sup_{\varepsilon>0} \frac{1}{\varepsilon} \int_0^\varepsilon |f(\bar{x}, x_n - s)| ds \right)^2 dx_n d\bar{x} \\ &\leq C_n \int_{\mathbf{R}^{n-1}} \int_{\mathbf{R}^1} |f(\bar{x}, x_n)|^2 dx_n d\bar{x} \\ &= C_n \int_{\mathbf{R}^n} |f(x)|^2 dx. \end{aligned}$$

The first inequality follows from the induction hypothesis; the second inequality follows from the Hardy-Littlewood maximal theorem. Hence, only (iii) remains to be proved.

Note that $(M_h f)^\wedge = m_h \hat{f}$ and $(N_h f)^\wedge = n_h \hat{f}$, where for each $\xi = (\bar{\xi}, \xi_n)$ in $\mathbf{R}^n$,

$$m_h(\xi) = \frac{1}{h} \int_h^{2h} e^{-2\pi i \bar{\xi} \cdot \gamma(t)} dt$$

and

$$n_h(\xi) = \frac{1}{h \gamma_n(h)} \int_h^{2h} \int_0^{\gamma_n(h)} e^{-2\pi i (\bar{\xi} \gamma(t) + \xi_n s)} ds dt.$$

Using Fubini's theorem and the Plancherel theorem

$$\begin{aligned} \int_{\mathbf{R}^n} |g(f)(x)|^2 dx &= \int_0^{N/2} \int_{\mathbf{R}^n} |M_h f(x) - N_h(x)|^2 dx \frac{dh}{h} \\ &= \int_0^{N/2} \int_{\mathbf{R}^n} |m_h(\xi) - n_h(\xi)|^2 |\hat{f}(\xi)|^2 d\xi \frac{dh}{h} \\ &= \int_{\mathbf{R}^n} \left(\int_0^{N/2} |m_h(\xi) - n_h(\xi)|^2 \frac{dh}{h} \right) |\hat{f}(\xi)|^2 d\xi. \end{aligned}$$

So to prove (iii) it is sufficient to prove (iv).

$$\forall \xi \in \mathbf{R}^n, \quad \int_0^{N/2} |m_h(\xi) - n_h(\xi)|^2 \frac{dh}{h} \leq C_n. \quad (\text{iv})$$

Define a to be the solution of $\gamma_n(2a) = 1/|\xi_n|$. Write the integral in (iv) as $\int_0^a + \int_a^{2a} + \int_{2a}^{N/2}$.

$$\int_a^{2a} |m_h(\xi) - n_h(\xi)|^2 \frac{dh}{h} \leq 4 \int_a^{2a} \frac{dh}{h} = 4 \log 2$$

since $|m_h(\xi)| \leq 1$ and $|n_h(\xi)| \leq 1$.

$$\begin{aligned} |m_h(\xi) - n_h(\xi)| &\leq \frac{1}{h\gamma_n(h)} \int_h^{2h} \int_0^{\gamma_n(h)} |e^{-2\pi i \xi_n s} - e^{-2\pi i \xi_n t}| ds dt \\ &\leq \frac{2\pi |\xi_n|}{h\gamma_n(h)} \int_h^{2h} \int_0^{\gamma_n(h)} (\gamma_n(t) - s) ds dt \leq 2\pi |\xi_n| \gamma_n(2h) \end{aligned}$$

so that

$$\begin{aligned} \int_0^a |m_h(\xi) - n_h(\xi)|^2 \frac{dh}{h} &\leq 4\pi^2 |\xi_n|^2 \int_0^a \gamma_n(2h)^2 \frac{dh}{h} \\ &\leq 4\pi^2 |\xi_n|^2 \gamma_n(2a) \int_0^{2a} \frac{\gamma_n(t)}{t} dt \\ &\leq 4\pi^2 |\xi_n|^2 \gamma_n(2a) \int_0^{2a} \gamma'_n(t) dt \\ &= 4\pi^2 |\xi_n|^2 \gamma_n(2a)^2 = 4\pi^2. \end{aligned}$$

$$\int_{2a}^{N/2} |m_h(\xi) - n_h(\xi)|^2 \frac{dh}{h} \leq 2 \int_{2a}^{N/2} |m_h(\xi)|^2 \frac{dh}{h} + 2 \int_{2a}^{N/2} |n_h(\xi)|^2 \frac{dh}{h}$$

and we estimate each of the last integrals separately.

Making use of Lemma 1 and the argument in the proof of Theorem 1 we see that

$$|m_h(\xi)| \leq C_n (|\xi_n| (D^n \gamma_n)(h) \alpha_1(h) \cdots \alpha_n(h))^{-1/n} h^{-1}.$$

Squaring and integrating shows that

$$\begin{aligned} \int_{2a}^{N/2} |m_h(\xi)|^2 \frac{dh}{h} &\leq C_n (|\xi_n| (D^n \gamma_n)(2a) \alpha_1(2a) \cdots \alpha_n(2a))^{-2/n} \int_{2a}^{\infty} \frac{dh}{h^3} \\ &= C_n (|\xi_n| (2a)^n D^n \gamma_n(2a) \alpha_1(2a) \cdots \alpha_n(2a))^{-2/n} \\ &\leq C_n (|\xi_n| \gamma_n(2a))^{-2/n} = C_n. \end{aligned}$$

The last inequality follows from Lemma 2.

$$|n_h(\xi)| \leq \frac{1}{\gamma_n(h)} \left| \int_0^{\gamma_n(h)} e^{-2\pi i \xi_n s} ds \right| \leq \frac{2}{|\xi_n| \gamma_n(h)}$$

follows from van der Corput's lemma. Hence

$$\begin{aligned} \int_{2a}^{N/2} |n_h(\xi)|^2 \frac{dh}{h} &\leq \frac{4}{|\xi_n|^2} \int_{2a}^{N/2} \frac{dh}{h [\gamma_n(h)]^2} \\ &\leq \frac{4}{|\xi_n|^2} \int_{2a}^{N/2} \frac{\gamma'_n(h) dh}{[\gamma_n(h)]^3} \leq \frac{2}{[|\xi_n| \gamma_n(2a)]^2} = 2. \end{aligned}$$

This completes the proof of (iv) and with it the proof of (i).

We have just proven (ii) in the case $p = 2$. The case $p = \infty$ is obvious with $C_{n,\infty} = 1$. The remaining case $2 < p < \infty$ follows from these results and the Marcinkiewicz interpolation theorem. Q.E.D.

Since Theorem 3 is obvious for continuous functions, the estimate in Theorem 2(i) and a standard argument imply Theorem 3.

It is known that if γ is a plane curve parametrized by arc length and the curvature of γ is increasing, then M is bounded on $L^2(\mathbb{R}^2)$. It is an interesting problem to determine similar geometric conditions on γ in higher dimensions which would guarantee that M is bounded on $L^2(\mathbb{R}^n)$.

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